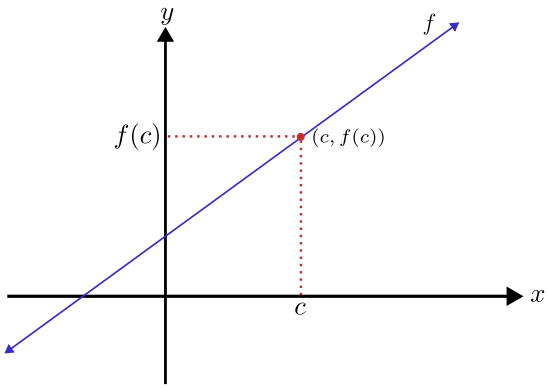
Section 2.3

As we saw in the last section, using tables or even graphs to evaluate limits can sometimes be surprisingly inefficient. Luckily, limits interact very nicely with algebraic operations.

Linear Functions

Graphically evaluating a limit $\lim_{x\to c} f(x)$ where f is a linear function is straightforward:



Theorem 2.3.1. For a linear function $f(x) = mx + b$, for any c we have

Example 2.3.2. Let $f(x) = 2x - \frac{1}{2}$ and $g(x) = 2$. Determine the following limits:

1. $\lim_{x\to 3} f(x)$

2. $\lim_{x\to -1} g(x)$

3. $\lim_{x\to 0} f(2)$

Theorem 2.3.3 (Limit Laws)**.** Let f and g be functions such that both $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. Let k be some number and $n > 0$ a positive integer. Then the following properties hold:

1. **Sum** $\lim_{x\to c}\big[f(x)+g(x)\big] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
2. **Constant multiple** $\lim_{x\to c} kf(x) = k\lim_{x\to c} f(x)$
3. **Difference** $\lim_{x\to c}\big[f(x)-g(x)\big] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$
4. **Product** $\lim_{x\to c}\big[f(x)g(x)\big] = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right)$
5. **Quotient** $\lim_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, provided $\lim_{x\to c} g(x) \neq 0$
6. **Powers** $\lim_{x\to c}\big[f(x)\big]^n = \left(\lim_{x\to c} f(x)\right)^n$
7. **Roots** $\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$

Example 2.3.4. Suppose $\lim_{x\to 2} f(x) = 5$, $\lim_{x\to 2} g(x) = 3$ and $\lim_{x\to 2} h(x) = -1$. Find

1. $\lim_{x\to 2}\big[2f(x) + g(x)h(x)\big]$

2. $\lim_{x\to 2}\sqrt{\dfrac{f(x)+h(x)}{4}}$

3. $\lim_{x\to 2}\dfrac{f(x)}{g(x)+3h(x)}$

Theorem 2.3.5 (Polynomial and Rational Function Limits)**.** Let p and q be polynomials. Then

1. **Limits of Polynomials** $\lim_{x\to c} p(x) = p(c)$
2. **Limits of Rational Functions** $\lim_{x\to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$, provided $q(c) \neq 0$

Example 2.3.6. Compute the following limits

1. $\lim_{x\to 4}(3x - 9)$

2. $\lim_{x\to 1}(2x^3 - 3x^2 + x - 5)$

3. $\lim_{x\to 2} \frac{x^2 - 5x - 2}{2x - 3}$

4. $\lim_{x\to 3} \frac{x^2 - 2x - 3}{x - 3}$

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Theorem 2.3.7 (Function Substitution)**.** Let f be a function defined for all x near c (possibly not defined at c). If g is a function such that $g(x) = f(x)$ for all x near c (possibly not at c), then if $\lim_{x\to c} g(x)$ exists

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x).$$

If $\lim_{x\to c} g(x)$ does not exist then $\lim_{x\to c} f(x)$ also does not exist.

Left-Sided Ver. If $g(x) = f(x)$ for all $x < c$ then

$$\lim_{x\to c^-} f(x) = \lim_{x\to c^-} g(x)$$

Right-Sided Ver. If $g(x) = f(x)$ for all $x > c$ then

$$\lim_{x\to c^+} f(x) = \lim_{x\to c^+} g(x)$$

Example 2.3.8. Find the following limits

1. $\lim_{x\to 3} \dfrac{x^2 - 2x - 3}{x - 3}$

2. $\lim_{x\to 1} \dfrac{x - 1}{\sqrt{x} - 1}$

3. $\lim_{x \to 1} \dfrac{x-1}{\sqrt{4x+5}-3}$

Substitution is also how we compute limits for piecewise defined functions. Let $f(x)$ be given by the rule

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < -1 \\ \sqrt{x+1} & \text{if } x \geq -1 \end{cases}$$

1. $\lim_{x \to -1^-} f(x)$

2. $\lim_{x \to -1^+} f(x)$

3. $\lim_{x \to -1} f(x)$

Theorem 2.3.9 (Squeeze Theorem)**.** Suppose f, g and h satisfy the inequality $f(x) \leq g(x) \leq h(x)$ for all x near c (except possibly at c). If

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L,$$

then

$$\lim_{x \to c} g(x) = L.$$

Example 2.3.10. Compute the following limits:

1. $\lim_{x \to 0} \sin x$

2. $\lim_{x \to 0} \cos x$